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Lie bialgebra contractions and quantum deformations of quasi-orthogonal algebras

A. Ballesteros[1], N. A. Gromov[2], F.J. Herranz, M.A. del Olmo and M. Santander

Departamento de Física Teórica, Universidad de Valladolid
E-47011, Valladolid, Spain

[1] *Departamento de Física, Universidad de Burgos*
Facultad de CYTA y Ciencias Químicas, E-09003, Burgos, Spain

[2] *In absence from Department of Mathematics, Komi Scientific Centre*
167000 Syktyvkar, Russia

Abstract

Lie bialgebra contractions are introduced and classified. A non-degenerate coboundary bialgebra structure is implemented into all pseudo-orthogonal algebras $so(p,q)$ starting from the one corresponding to $so(N+1)$. It allows to introduce a set of Lie bialgebra contractions which leads to Lie bialgebras of quasi-orthogonal algebras. This construction is explicitly given for the cases $N = 2, 3, 4$. All Lie bialgebra contractions studied in this paper define Hopf algebra contractions for the Drinfel'd-Jimbo deformations $U_z so(p,q)$. They are explicitly used to generate new non-semisimple quantum algebras as it is the case for the Euclidean, Poincaré and Galilean algebras.

1 Introduction

Several non-semisimple Lie groups play an important role in Physics, for instance, the Poincaré and Galilei ones; they can be got starting from semisimple groups by means of a sequence of contractions [1]. The current interest in quantum deformations of Lie algebras raised the extension of the idea of contraction from Lie algebras to their quantum analogues taking into account the bearing of the contraction on the deformation parameter [2]. In this way, the number of different possible contractions to perform on a given algebra increases significantly, as well as the difficulties encountered to analyse the convergency properties of each of them.

On the other hand, the main underlying structure of a quantum algebra is the Lie bialgebra that gives the first order term in the deformation [3, 4]. Higher order terms can be, in principle, obtained by a consistency method, and the classification problem for quantum deformations is rather simplified by taking into account this fact [5]. The aim of this paper is to show that the information related to a given Hopf algebra contraction can be extracted with much less effort from the underlying Lie bialgebra, thus providing a simplified approach to a (constructive) classification of Hopf algebra contractions.

This program is developed here for the non-degenerate (or standard) coboundary Lie bialgebras linked to the quantum orthogonal algebras $so(N+1)$; they are generated by an r–matrix which satisfies the modified classical Yang–Baxter equation (YBE). Firstly, we endow the pseudo-orthogonal algebras $so(p,q)$ with this bialgebra structure and afterwards we study a set of Lie bialgebra contractions which provides quasi-orthogonal bialgebras.

The main concepts about (coboundary) Lie bialgebras and their contractions are established in section 2. The cases $N = 2, 3, 4$ are fully analysed in sections 3, 4 and 5, respectively. We explicitly study all the possible choices of the behaviour of the deformation parameter under the chosen contractions, and classify the divergencies they produce both in the classical r–matrix and in the bialgebra mapping δ. We study separately these objects, since starting from a coboundary Lie bialgebra a contraction can either produce a coboundary bialgebra (both r and δ do not diverge) or a (right) bialgebra that is not a coboundary (r diverges but δ is well defined). Therefore, a separate analysis of the behaviour of r and δ under contraction makes more clear the link between non-semisimple algebras and non-coboundary structures.

2 Lie bialgebras and their contractions

Firstly, we present the basic concepts that we shall need in order to introduce Lie bialgebra contractions. Basic facts about quantum algebras can be found in references [3, 4, 6, 7, 8, 9, 10].

2.1 Lie bialgebras

Definition 2.1. *A Lie bialgebra (g, η) is a Lie algebra g endowed with a cocommutator $\eta : g \to g \otimes g$ such that*

i) η is a 1–cocycle, i.e.,

$$\eta([x, y]) = [\eta(x),\, 1 \otimes y + y \otimes 1] + [1 \otimes x + x \otimes 1,\, \eta(y)], \quad \forall x, y \in g. \tag{2.1}$$

ii) The dual map $\eta^* : g^* \otimes g^* \to g^*$ is a Lie bracket on g^*.

Definition 2.2. *A Lie bialgebra (g, η) is called a coboundary bialgebra if there exists an element $\rho \in g \otimes g$ called r–matrix, such that*

$$\eta(x) = [1 \otimes x + x \otimes 1,\, \rho], \qquad \forall x \in g. \tag{2.2}$$

It can be easily shown that the map (2.2) defined by means of an arbitrary ρ is a Lie bialgebra if and only if the symmetric part of ρ is a g–invariant element of $g \otimes g$ and the Schouten bracket

$$[\rho, \rho] := [\rho_{12}, \rho_{13}] + [\rho_{12}, \rho_{23}] + [\rho_{13}, \rho_{23}] \tag{2.3}$$

is a g–invariant element of $g \otimes g \otimes g$ (ρ fulfills the modified classical YBE: $ad_g[\rho, \rho] = 0$). Here $\rho_{12} = \rho \otimes 1$, and the same convention is taken for ρ_{13} and ρ_{23}. In fact, we shall consider skew-symmetric r–matrices, and we shall denote coboundary Lie bialgebras in the form

$(g, \eta(\rho))$.

Definition 2.3. *Two Lie bialgebras (g, η_1) and (g, η_2) are said to be equivalent if there exists a Lie algebra automorphism D of g such that $\eta_2 = (D^{-1} \otimes D^{-1}) \circ \eta_1 \circ D$.*

As a consequence, two r–matrices ρ_1 and ρ_2 will be considered as equivalent if the bialgebras $(g, \eta_1(\rho_1))$ and $(g, \eta_2(\rho_2))$ generated by them are equivalent according to Def. 2.3.

Let us recall that a quantum universal enveloping algebra (QUEA) $A = U_z g$ is a quantization of a Lie bialgebra (g, η) in the following sense: if we write the coproduct $\Delta : A \to A \otimes A$ as a formal power series in the deformation parameter z

$$\Delta = \sum_{k=0}^{\infty} \Delta_{(k)} = \sum_{k=0}^{\infty} z^k \eta_{(k)}, \tag{2.4}$$

the classical limit condition $A/zA \equiv Ug$ implies that the mapping defined by

$$\eta := (\Delta_{(1)} - \sigma \circ \Delta_{(1)}) \mod z, \tag{2.5}$$

is a Lie bialgebra mapping (here, $\sigma(a \otimes b) := b \otimes a$). If $\Delta_{(1)}$ is skew-symmetric, we have that $\eta = 2\eta_{(1)}$. Moreover, higher order terms in the coproduct (2.4) can be reconstructed in terms of the first order one by solving the coassociativity condition $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ order by order in z, which reads:

$$\sum_{n=0}^{k} (\Delta_{(n)} \otimes 1 - 1 \otimes \Delta_{(n)})\Delta_{(k-n)} = 0. \tag{2.6}$$

2.2 Contraction of Lie bialgebras

We first recall the definition of Lie algebra contraction in a general setting [11, 12]. Hereafter, g will be a finite dimensional Lie algebra.

Definition 2.4. *A Lie algebra g' is a contraction of another Lie algebra g (with the same underlying vector space V) if there exists a oneparametric family of Lie algebra automorphisms*

$\phi_\varepsilon : g \to g$ such that the limit $\lim_{\varepsilon\to 0} \phi_\varepsilon^{-1}[\phi_\varepsilon(X), \phi_\varepsilon(Y)]$ of the Lie bracket in g gives the Lie bracket $[X,Y]'$ in g'.

The mappings ϕ_ε can be interpreted, as long as $\varepsilon \neq 0$, as an embedding of the Lie generators of g' within g. Relations among the former ones are computed as embedded in g, and turned back to g' by using ϕ_ε^{-1} and then making the limit $\varepsilon \to 0$. We will assume that the automorphisms

ϕ_ε have a polynomial dependence on ε, as it is the case in most physical examples of contractions. The contraction of the Lie bracket $[\,,\,] : g \otimes g \to g$ so defined can be generalized to the bialgebra mapping $\eta : g \to g \otimes g$ and to the r–matrix $\rho \in g \otimes g$ as follows:

Proposition 2.5. *Let (g, η) be a Lie bialgebra and let g' be a contraction of g defined by the mappings ϕ_ε. If n is any real number such that the limit*

$$\eta' := \lim_{\varepsilon\to 0} \varepsilon^n (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1}) \circ \eta \circ \phi_\varepsilon, \tag{2.7}$$

exists, then (g', η') is a Lie bialgebra. Furthermore, there exists a single minimal fixed value f_0 of n such that for $n \geq f_0$ the limit (2.7) exists and for $n > f_0$ the limit is zero.

For a given family of automorphisms ϕ_ε defining the contraction of the Lie algebra $g \to g'$, we get from Prop. 2.5 a family of Lie bialgebras (g', η') parametrized by the real number $n \geq f_0$, which can be naturally considered as contractions of (g, η). Note however that η' is a non-trivial cocommutator only when $n = f_0$. Therefore, the following definitions make sense:

Definition 2.6. *The Lie bialgebra (g', η') is said to be a Lie bialgebra contraction (LBC) of (g, η) if there exists a contraction from g to g' described by a family ϕ_ε of Lie algebra automorphisms and a number n such that η' is given by the limit (2.7). We denote such a contraction by the pair (ϕ_ε, n).*

Definition 2.7. *The minimal value f_0 of n will be called the fundamental contraction constant of the Lie bialgebra (g, η) associated to the family ϕ_ε.*

Definition 2.8. *The LBC with the minimal value f_0 of n (ϕ_ε, f_0) is said to be the fundamental LBC of (g, η) associated to the mappings ϕ_ε.*

The preceding discussion applies to any Lie bialgebra, not necessarily a coboundary one. But for a coboundary Lie bialgebra it is rather natural to study directly the contraction of the associated r–matrix. More specifically:

Proposition 2.9. *Let $(g, \eta(\rho))$ be a coboundary Lie bialgebra and let g' be a*

contraction of g defined by the mappings ϕ_ε. If n is a real number such that the limit

$$\rho' := \lim_{\varepsilon\to 0} \varepsilon^n (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1})(\rho), \tag{2.8}$$

exists, then $(g', \eta'(\rho'))$ is a coboundary Lie bialgebra where η' is obtained by applying (2.2). Moreover, there exists a single minimal fixed value c_0 of n such that for $n \geq c_0$ the limit (2.8) exists and for $n > c_0$ the limit is zero.

Obviously, the coboundary bialgebra $(g', \eta'(\rho'))$ is a contraction of $(g, \eta(\rho))$ in the sense of Def. 2.6. On the other hand, Defs. 2.7 and 2.8 can be suitably modified leading to:

Definition 2.10. *The minimal value c_0 of n will be called the coboundary contraction constant of the Lie bialgebra $(g, \eta(\rho))$ associated to the family ϕ_ε.*

Definition 2.11. *The LBC with the minimal value c_0 of n (ϕ_ε, c_0) is said to be the coboundary LBC of $(g, \eta(\rho))$ associated to the mappings ϕ_ε.*

It is clear that $f_0 \leq c_0$ always hold (cfr. Prop. 2.5); the important point to be emphasized here is that both limits (2.7) and (2.8), being of a different nature, should be analysed separately and do not necessarily lead to equal values for the fundamental and coboundary contraction constants of a given coboundary bialgebra. When $f_0 = c_0$, the contraction (ϕ_ε, n) with $n = c_0 = f_0$ is called a *"fundamental coboundary LBC"* (both the contracted cocommutator and r–matrix are not trivial). If $f_0 < c_0$ the LBC with coboundary contraction constant c_0 is *not* a fundamental one, because the contracted cocommutator is zero and r' will be either trivial or equivalent to the trivial one; hence, in this situation the LBC with fundamental contraction constant f_0 is *not* a coboundary one.

In general, it is necessary to allow for n in order to ensure the convergency of the limit (2.7). However, if we consider Lie bialgebras as generating objects for quantum algebras, this fact can be interpreted in a different way (as a kind of "renormalization") by including the deformation parameter z within the Lie cocommutator, that we shall denote as $\delta(X) := z\,\eta(X)$. This can be done provided we are able to find a homomorphism D giving the equivalence between δ and η (see Def. 2.3.; the mere multiplication of all generators of g by z gives the simplest form of D). In this sense, the renormalization factor ε^n can be associated to a transformation law of the deformation parameter

$$w = \phi_\varepsilon(z) := \varepsilon^{-n} z, \tag{2.9}$$

where w would be the new (contracted) deformation parameter. The contracted structure can be now defined without any factor ε^n:

$$\begin{aligned} \delta' &:= \lim_{\varepsilon\to 0} (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1}) \circ \delta \circ \phi_\varepsilon = \lim_{\varepsilon\to 0} z\, (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1}) \circ \eta \circ \phi_\varepsilon \\ &= \lim_{\varepsilon\to 0} w\, \varepsilon^n\, (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1}) \circ \eta \circ \phi_\varepsilon = w\, \eta'. \end{aligned} \tag{2.10}$$

Obviously, if the Lie bialgebra (g, η) is a coboundary one generated by ρ, then (g, δ) will be also a coboundary Lie bialgebra generated by $r := z\rho$. The very same

extension of the action of the mappings ϕ_ε given by (2.9) leads to the following modification of (2.8):

$$r' := \lim_{\varepsilon\to 0} (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1})(r) = \lim_{\varepsilon\to 0} w\, \varepsilon^n (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1})(\rho) = w\, \rho'. \tag{2.11}$$

This behaviour of the deformation parameter was soon discovered as a condition to obtain non-semisimple quantum algebras by contraction [2]; moreover it provided a dimensional interpretation of the quantum parameter [13]. At this respect, the LBC framework we have just developed arises as relevant, since for all the cases that have been worked out, the existence of a fundamental LBC (ϕ_ε, f_0) is a *sufficient* condition for the convergency of the entire Hopf structure under the Hopf algebra contraction defined by

$$\Delta' := \lim_{\varepsilon\to 0} (\phi_\varepsilon^{-1} \otimes \phi_\varepsilon^{-1}) \circ \Delta \circ \phi_\varepsilon. \tag{2.12}$$

This fact seems to be rather general: for a given Lie algebra contraction, the fundamental LBC defines in a unique way the transformation law of z (the change of generators is alocady given by the classical contraction ϕ_ε) and ensures the first order deformation to be well-defined. Afterwards, the coassociativity constrain enters, and propagates the convergency of this first order term to higher order ones.

In the sequel, this result is shown to be very useful, since the systematic classification of LBC's allows us to find some new quantum non-semisimple algebras by considering all possible transformations of z that still define right LBC's. Noncoboundary structures will appear in a natural way in this context, and a global overview of all possible contractions of a given Hopf algebra is obtained.

2.3 Quasi-orthogonal algebras as contractions from $so(p,q)$ algebras

The purpose of this paper is to analyse the contraction scheme sketched in the previous section for a specific particular set of Lie bialgebras, which include pseudo-orthogonal and quasi-orthogonal bialgebras. For any $N = 2, 3, \ldots$, let us consider the real Lie algebra with $N(N+1)/2$ generators J_{ab}, $a < b; (a, b = 0, 1, \ldots, N)$, and commutation relations:

$$[J_{ab}, J_{ac}] = \kappa_{ab} J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = \kappa_{bc} J_{ab}, \quad a < b < c, \tag{2.13}$$

(those commutators involving four different indices are equal to zero), where the structure constants κ_{ab} depend on N real numbers $\kappa_1, \ldots, \kappa_N$:

$$\kappa_{ab} = \prod_{s=a+1}^{b} \kappa_s, \qquad a < b. \tag{2.14}$$

These algebras are called Cayley–Klein (CK) algebras [14, 15, 16, 17]. We will denote them by $g_{(\kappa_1,\ldots,\kappa_N)}$. Upon rescaling of generators, each κ_i can be reduced separately to 1, 0 or -1, so the system of CK algebras includes 3^N Lie algebras.

When all $\kappa_i \neq 0$, it is easy to identify (2.13) with the pseudo-orthogonal algebra $so(p,q)$, $(p+q=N+1)$ which leaves invariant the bilinear form

$$\Lambda^{(0)} = \mathrm{diag}\,(1, \kappa_{01}, \kappa_{02}, \ldots, \kappa_{0N}) = \mathrm{diag}\,(1, \kappa_1, \kappa_1\kappa_2, \ldots, \kappa_1 \cdots \kappa_N). \qquad (2.15)$$

Thus for all $\kappa_i = 1$ we recover the $so(N+1)$ algebra. But the pseudo-orthogonal algebras are far from exhausting the family of CK algebras. For instance, if $\kappa_1 = 0$ with all remaining $\kappa_i \neq 0$ the CK algebra has a semidirect sum structure: $g_{(0,\kappa_2,\ldots,\kappa_N)} = t_N \odot so(p',q')$ $(p'+q'=N)$, where $t_N = \langle J_{0i} \rangle$ $(i=1,\ldots,N)$ is an Abelian subalgebra and the $so(p',q') = \langle J_{ij} \rangle$ $(i,j=1,\ldots,N)$ subalgebra leaves invariant the bilinear form

$$\Lambda^{(1)} = \mathrm{diag}\,(1, \kappa_{12}, \kappa_{13}, \ldots, \kappa_{1N}) = \mathrm{diag}\,(1, \kappa_2, \kappa_2\kappa_3, \ldots, \kappa_2 \cdots \kappa_N). \qquad (2.16)$$

Therefore, the N–dimensional Euclidean algebra $iso(N)$ corresponds to $g_{(0,1,\ldots,1)}$ while the N–dimensional Poincaré algebra $iso(N-1,1)$ appears several times: $g_{(0,-1,1,\ldots,1)}$, $g_{(0,-1,-1,1,\ldots,1)}$, $g_{(0,1,-1,\ldots,-1,1)}$, $g_{(0,1,\ldots,1,-1)}$, etc.

Within the set of CK Lie algebras $g_{(\kappa_1,\ldots,\kappa_N)}$ there exists a family of N Inönü–Wigner (IW) contractions

$$\phi_{\varepsilon_i}(J_{ab}) = \left\{ \begin{array}{cl} \varepsilon_i J_{ab} & \text{if either } a \text{ or } b \in \{0,1,\ldots,i-1\} \\ J_{ab} & \text{otherwise} \end{array} \right. , \quad i=1,\ldots,N; \qquad (2.17)$$

that can be applied to any algebra in the CK family. When the contraction defined by ϕ_{ε_i} is performed on $g_{(\kappa_1,\ldots,\kappa_i,\ldots,\kappa_N)}$ we get $g_{(\kappa_1,\ldots,\kappa_{i-1},0,\kappa_{i+1},\ldots,\kappa_N)}$ as contracted algebra. The set of contractions just defined have a commutative character; we can apply to a given Lie algebra as many contractions of the kind (2.17) as desired and the order is inmaterial for the result.

In particular, the contraction ϕ_{ε_1} leads to inhomogeneous algebras and it is called *affine contraction* since the space $G_{(0,\kappa_2,\ldots,\kappa_N)}/G_{(\kappa_2,\ldots,\kappa_N)}$ is a flat affine space. Another interesting example is the N–dimensional Galilean algebra that corresponds to $g_{(0,0,1,\ldots,1)}$; it can be got from either the Euclidean or Poincaré algebras by means of the contraction ϕ_{ε_2}. A explicit study of CK algebras for $N=2,3,4$ are given in refs. [18], [19] and [13], respectively.

The CK algebras can be viewed from another point of view. Let us now consider the algebra $g_{(1,\ldots,1)} \equiv so(N+1)$. Its non-zero Lie brackets are

$$[\widetilde{J}_{ab}, \widetilde{J}_{ac}] = \widetilde{J}_{bc}, \quad [\widetilde{J}_{ab}, \widetilde{J}_{bc}] = -\widetilde{J}_{ac}, \quad [\widetilde{J}_{ac}, \widetilde{J}_{bc}] = \widetilde{J}_{ab}, \quad a<b<c. \qquad (2.18)$$

If we define the following set of N mappings $\tilde{\phi}_{\kappa_i}$ $(i=1,\ldots,N)$:

$$\tilde{\phi}_{\kappa_i}(\widetilde{J}_{ab}) = \left\{ \begin{array}{cl} \sqrt{\kappa_i}\,\widetilde{J}_{ab} & \text{if either } a \text{ or } b \in \{0,1,\ldots,i-1\} \\ \widetilde{J}_{ab} & \text{otherwise} \end{array} \right. , \quad \forall \kappa_i \neq 0; \qquad (2.19)$$

the composition of all these mappings gives rise to the formal transformation

[20]:

$$J_{ab} := \Phi_N(\widetilde{J}_{ab}) = \tilde{\phi}_{\kappa_1} \circ \cdots \circ \tilde{\phi}_{\kappa_N}(\widetilde{J}_{ab}) = \sqrt{\kappa_{ab}}\,\widetilde{J}_{ab}, \quad \forall \kappa_{ab} \neq 0. \tag{2.20}$$

It is easy to check that if the generators $\widetilde{J}_{ab}$ close the $so(N+1)$ algebra (2.18), then the transformed generators J_{ab} (all κ_i are different from zero) close the CK algebra (2.13). Thus, strictly speaking, the transformation (2.20) relates $so(N+1)$ with the whole set of pseudo-orthogonal algebras $so(p,q)$. Moreover, if suitably understood as a limiting procedure (whenever some $\kappa_i = 0$ a limit $\sqrt{|\kappa_i|} \to 0$ should be made in the family $\tilde{\phi}_{\kappa_i}$), then (2.19) and (2.20) can be formally applied even when κ_i are allowed to be zero. In this case, Φ_N goes from the $so(N+1)$ algebra to the general CK algebra with no restrictions as to the vanishing of the κ_i constants. In this sense, the mapping (2.19) is equivalent, when some $\kappa_i = 0$, to the combination of a standard Weyl unitary trick and an IW contraction (2.17), the contraction parameter being

$$\varepsilon_i := \sqrt{|\kappa_i|}, \quad i = 1, \ldots, N. \tag{2.21}$$

Therefore, we can profit from the transformation (2.20) in a double sense: firstly, *any* expression for the general CK algebra $g_{(\kappa_1,\ldots\kappa_N)}$ can be got starting from the one corresponding to $so(N+1)$. Secondly, the scheme so obtained automatically includes a set of IW contractions implicitly expressed in terms of the $\tilde{\phi}_{\kappa_i}$ transformations, as given in (2.17) and (2.21).

We apply this device to the contractions of Lie bialgebras according to the ideas introduced in Sect. 2.2. The transformation (2.20) should be augmented with the appropriate transformation $\Phi_N(z)$ of the deformation parameter, in order to obtain a set of LBC's wich can be applied to *any* bialgebra for the CK algebras. This last step is only required as far as contractions are involved, because the mere substitution (2.20) when applied to a $so(N+1)$ bialgebra would produce a well defined coboundary bialgebra for all pseudo-orthogonal algebras $so(p,q)$. This procedure gives us the coboundary and the fundamental contraction constants that classify the LBC's.

Since we are going to work simultaneously with this commuting set of classical contractions $(\phi_{\varepsilon_1}, \ldots, \phi_{\varepsilon_N})$ that will originate Lie bialgebra ones, it will be useful to introduce the *"sets of fundamental bialgebras"* , that will be denoted by $(g_{(\kappa_1,\ldots,\kappa_N)}, \delta^{(f_{01},\ldots,f_{0N})})$. They will be the result of the fundamental LBC's $(\phi_{\varepsilon_1}, f_{01})$, $\ldots$, $(\phi_{\varepsilon_N}, f_{0N})$ acting on a given original CK bialgebra $(g_{(\kappa_1,\ldots,\kappa_N)}, \delta)$ and give a precise Lie bialgebra for each of the CK algebras. Hence in this case, the transformation of the deformation parameter (2.9) will be

$$w = \Phi_N(z) := \varepsilon_1^{-f_{01}} \ldots \varepsilon_N^{-f_{0N}} z. \tag{2.22}$$

These sets will underly the quantum algebras denoted by $(U_w g_{(\kappa_1,\ldots,\kappa_N)}, \Delta)$, which are deformations of the classical algebra $g_{(\kappa_1,\ldots,\kappa_N)}$ in which $\delta^{(f_{01},\ldots,f_{0N})}$ gives the first order term in z within the coproduct Δ. These quantum algebras are straightforwardly obtained by applying Φ_N to $so(N+1)$ in the form (2.12).

3 The so(3) case

Let us consider the $so(3)$ Lie algebra generated by $\{\widetilde{J}_{01}, \widetilde{J}_{02}, \widetilde{J}_{12}\}$. Its standard (uniparametric) Drinfel'd–Jimbo deformation is given by the following coproduct Δ_{02} and commutators $[\ ,\ ]_{02}$:

$$\begin{aligned}
\Delta_{02}(\widetilde{J}_{02}) &= 1 \otimes \widetilde{J}_{02} + \widetilde{J}_{02} \otimes 1, \\
\Delta_{02}(\widetilde{J}_{01}) &= e^{-\frac{z}{2}\widetilde{J}_{02}} \otimes \widetilde{J}_{01} + \widetilde{J}_{01} \otimes e^{\frac{z}{2}\widetilde{J}_{02}}, \\
\Delta_{02}(\widetilde{J}_{12}) &= e^{-\frac{z}{2}\widetilde{J}_{02}} \otimes \widetilde{J}_{12} + \widetilde{J}_{12} \otimes e^{\frac{z}{2}\widetilde{J}_{02}},
\end{aligned} \tag{3.1}$$

$$[\widetilde{J}_{12}, \widetilde{J}_{01}]_{02} = \frac{\sinh z\widetilde{J}_{02}}{z}, \qquad [\widetilde{J}_{12}, \widetilde{J}_{02}]_{02} = -\widetilde{J}_{01}, \qquad [\widetilde{J}_{01}, \widetilde{J}_{02}]_{02} = \widetilde{J}_{12}. \tag{3.2}$$

The label $\{02\}$ in the coproduct and commutation relations reminds the choice of the primitive generator.

3.1 $so(3)$ bialgebras and their contractions

Obviously, we could have written two more (completely equivalent) structures by choosing either $\widetilde{J}_{01}$ or $\widetilde{J}_{12}$ as the primitive elements. Explicitly, these three possibilities are characterized by the following r–matrices:

$$r_{01} = z(\widetilde{J}_{02} \wedge \widetilde{J}_{12}), \quad r_{02} = z(\widetilde{J}_{12} \wedge \widetilde{J}_{01}), \quad r_{12} = z(\widetilde{J}_{01} \wedge \widetilde{J}_{02}). \tag{3.3}$$

Thus, the associated coboundary bialgebras are given, respectively, by

$$\delta_{01}(\widetilde{J}_{12}) = z(\widetilde{J}_{12} \wedge \widetilde{J}_{01}), \quad \delta_{01}(\widetilde{J}_{01}) = 0, \quad \delta_{01}(\widetilde{J}_{02}) = z(\widetilde{J}_{02} \wedge \widetilde{J}_{01}); \tag{3.4}$$

$$\delta_{02}(\widetilde{J}_{12}) = z(\widetilde{J}_{12} \wedge \widetilde{J}_{02}), \quad \delta_{02}(\widetilde{J}_{01}) = z(\widetilde{J}_{01} \wedge \widetilde{J}_{02}), \quad \delta_{02}(\widetilde{J}_{02}) = 0; \tag{3.5}$$

$$\delta_{12}(\widetilde{J}_{12}) = 0, \quad \delta_{12}(\widetilde{J}_{01}) = z(\widetilde{J}_{01} \wedge \widetilde{J}_{12}), \quad \delta_{12}(\widetilde{J}_{02}) = z(\widetilde{J}_{02} \wedge \widetilde{J}_{12}). \tag{3.6}$$

These structures are related among themselves by means of appropriate permutations $\pi \in S_3$ on the set of indices $\{0, 1, 2\}$:

$$r_{01} = \pi_{(12)}(r_{02}), \qquad r_{12} = \pi_{(01)}(r_{02}), \tag{3.7}$$

and the same for the cocommutators, where permutations are denoted in cycle notation, so that $\pi_{(12)}$ is the 2–cycle $(1\,2)$ on the three indices $\{0, 1, 2\}$.

Within $so(3)$, all these Lie bialgebras are equivalent (the intertwining operator D of Def. 2.3 is just defined by the permutation). However, this equivalence can be broken when a LBC is performed. If we fix a given set of Lie algebra contractions (the mappings ϕ_{ε_i}) we find that, in general, the associated LBC's depend on the coboundary Lie bialgebras $(so(3), \delta(r_{ab}))$ we started with. In other words: in $so(3)$ we have only one (non-degenerate) bialgebra, but for non-semisimple algebras that can be obtained as contractions of $so(3)$, in general there exist more than one (non-equivalent) Lie bialgebra.

Explicitly, we apply the formal transformation Φ_2 (2.20) to the r–matrices and to the cocommutators of $so(3)$, obtaining in this way

coboundary Lie bialgebras for the $so(3)$ and $so(2,1)$ algebras included in the CK algebra $g_{(\kappa_1,\kappa_2)}$:

$$r'_{01} = (\Phi_2^{-1}\otimes\Phi_2^{-1})(r_{01}) = \frac{\Phi_2^{-1}(w)}{\sqrt{\kappa_1}\kappa_2}(J_{02}\wedge J_{12}), \tag{3.8}$$

$$r'_{02} = (\Phi_2^{-1}\otimes\Phi_2^{-1})(r_{02}) = \frac{\Phi_2^{-1}(w)}{\sqrt{\kappa_1\kappa_2}}(J_{12}\wedge J_{01}), \tag{3.9}$$

$$r'_{12} = (\Phi_2^{-1}\otimes\Phi_2^{-1})(r_{12}) = \frac{\Phi_2^{-1}(w)}{\kappa_1\sqrt{\kappa_2}}(J_{01}\wedge J_{02}); \tag{3.10}$$

$$\delta'_{01}(J_{12}) = \frac{\Phi_2^{-1}(w)}{\sqrt{\kappa_1}}(J_{12}\wedge J_{01}), \quad \delta'_{01}(J_{01}) = 0, \quad \delta'_{01}(J_{02}) = \frac{\Phi_2^{-1}(w)}{\sqrt{\kappa_1}}(J_{02}\wedge J_{01}); \tag{3.11}$$

$$\delta'_{02}(J_{12}) = \frac{\Phi_2^{-1}(w)}{\sqrt{\kappa_1\kappa_2}}(J_{12}\wedge J_{02}), \quad \delta'_{02}(J_{01}) = \frac{\Phi_2^{-1}(w)}{\sqrt{\kappa_1\kappa_2}}(J_{01}\wedge J_{02}), \quad \delta'_{02}(J_{02}) = 0; \tag{3.12}$$

$$\delta'_{12}(J_{12}) = 0, \quad \delta'_{12}(J_{01}) = \frac{\Phi_2^{-1}(w)}{\sqrt{\kappa_2}}(J_{01}\wedge J_{12}), \quad \delta'_{12}(J_{02}) = \frac{\Phi_2^{-1}(w)}{\sqrt{\kappa_2}}(J_{02}\wedge J_{12}). \tag{3.13}$$

The expresions (3.8–3.13) contain all the information needed in order to classify the LBC's we are dealing with. Let us explain one example. If we choose $z = \Phi_2^{-1}(w) := \sqrt{\kappa_1}\,\kappa_2\,w$, the limits $\kappa_i \to 0$ of r'_{01} (see (3.8)) will exist, and the r–matrix $r'_{01} = w(J_{02}\wedge J_{12})$ will provide a coboundary bialgebra structure for all the CK algebras $g_{(\kappa_1,\kappa_2)}$. In this way, the values for the coboundary contraction constants c_0 linked to $(g_{(\kappa_1,\kappa_2)},\delta'_{01})$ can be obtained. At this moment, it is important to recall that strictly speaking the contraction parameter which goes to zero in a contraction limit is indeed

$\varepsilon_i = \sqrt{|\kappa_i|}$, hence in this example the values of c_0 are 1 for ϕ_{ε_1} and 2 for ϕ_{ε_2}. The same method applied on cocommutators δ'_{01} gives the fundamental contraction constants f_0: 1 for ϕ_{ε_1} and 0 for ϕ_{ε_2}. The final result is summarized in the following theorem:

Theorem 3.1. *Let r'_{ab} be an r–matrix of $g_{(\kappa_1,\kappa_2)}$ with $\kappa_1,\kappa_2 \neq 0$ and ϕ_{ε_i} the family of automorphisms describing the classical contraction $\kappa_i \to 0$. The fundamental and the coboundary LBC's are defined by the LBC constants f_0 and c_0 given in table I.*

Table I. LBC constants for $g_{(\kappa_1,\kappa_2)}$.

Lie bialgebra	ϕ_{ε_1}		ϕ_{ε_2}	
	f_0	c_0	f_0	c_0
$(g_{(\kappa_1,\kappa_2)},\delta'_{01})$	1	1	0	2
$(g_{(\kappa_1,\kappa_2)},\delta'_{02})$	1	1	1	1
$(g_{(\kappa_1,\kappa_2)},\delta'_{12})$	0	2	1	1

This approach gives an unified overview of the many contracted structures in a very condensed way. Some of these structures are new and others appear in the literature. For instance, let us study the affine contraction ϕ_{ε_1} which carries $so(3)$ into the Euclidean algebra. When we choose not to transform the deformation parameter under this contraction (LBC $(\phi_{\varepsilon_1}, 0)$) we have $z = w(\kappa_1)^0 = w$, and r'_{12} does not exist, but the contracted cocommutator δ'_{12} does. This case corresponds to the non-coboundary bialgebra underlying Vaksman-Korogodski $e(2)_q$ algebra [21] (with $\kappa_2 = 1$), which is in turn dual to Woronowicz's $E(2)_q$ group [22]. In this case, when defined, LBC's which do not change the deformation parameter ($z = w$) give *only* non-coboundary Lie bialgebras, as the contractions studied in [23]. On the contrary, if we allow the transformation of the parameter z corresponding to the LBC $(\phi_{\kappa_1}, 2)$, then we find that r'_{12} exists under $\kappa_1 \to 0$, but δ'_{12} becomes trivial in this circumstance (and, hence, the Lie bialgebra is trivial): as we shall see immediately, the Hopf algebra contraction so defined leads to a non-deformed structure.

3.2 Hopf algebra contractions of $U_z so(3)$

Table I provides three sets of fundamental bialgebras: $(g_{(\kappa_1,\kappa_2)}, \delta_{01}^{(1,0)})$,

$(g_{(\kappa_1,\kappa_2)}, \delta_{02}^{(1,1)})$ and

$(g_{(\kappa_1,\kappa_2)}, \delta_{12}^{(0,1)})$. This means that, for each of the nine CK algebras with fixed κ_1 and κ_2, we have *three* –not necessarily equivalent– Lie bialgebra structures. These three sets of fundamental LBC's define three sets of Hopf algebra contractions that give rise to three quantum deformations of the CK algebra $g_{(\kappa_1,\kappa_2)}$:

(1) The fundamental LBC's defined by $(g_{(\kappa_1,\kappa_2)}, \delta_{02}^{(1,1)})$ leads to the choice of the transformation law for the deformation parameter ($z = \sqrt{\kappa_1\kappa_2}w$) taken originally for the CK scheme in [18]. In this case, the quantum algebra $U_w g_{(\kappa_1,\kappa_2)}$ has coproduct Δ_{02} (3.1) and the following commutation rules

$$[J_{12}, J_{01}]_{02} = \frac{\sinh wJ_{02}}{w}, \qquad [J_{12}, J_{02}]_{02} = -\kappa_2 J_{01}, \qquad [J_{01}, J_{02}]_{02} = \kappa_1 J_{12}. \quad (3.14)$$

The Hopf algebras $(U_w g_{(0,1)}, \Delta_{02})$, $(U_w g_{(0,-1)}, \Delta_{02})$ and $(U_w g_{(0,0)}, \Delta_{02})$ correspond, in this order, to quantum deformations of the Euclidean, Poincaré and Galilean (Heisenberg) algebras and have been studied in [24], [25] and [26], respectively. Note also that $(g_{(\kappa_1,\kappa_2)}, \delta_{02}^{(1,1)})$ is the only set of fundamental bialgebras which are coboundary as well.

(2) The Hopf algebra that quantizes $(g_{(\kappa_1,\kappa_2)}, \delta_{01}^{(1,0)})$ is obtained in two steps: firstly, we apply the permutation $\pi_{(12)}$ on (3.1–3.2). Afterwards, the LBC's associated to this case are evaluated obtaining:

$$\begin{aligned} \Delta_{01}(J_{01}) &= 1 \otimes J_{01} + J_{01} \otimes 1, \\ \Delta_{01}(J_{02}) &= e^{-\frac{w}{2}J_{01}} \otimes J_{02} + J_{02} \otimes e^{\frac{w}{2}J_{01}}, \\ \Delta_{01}(J_{12}) &= e^{-\frac{w}{2}J_{01}} \otimes J_{12} + J_{12} \otimes e^{\frac{w}{2}J_{01}}, \end{aligned} \quad (3.15)$$

$$[J_{12}, J_{01}]_{01} = J_{02}, \qquad [J_{12}, J_{02}]_{01} = -\kappa_2 \frac{\sinh w J_{01}}{w}, \qquad [J_{01}, J_{02}]_{01} = \kappa_1 J_{12}. \tag{3.16}$$

Now, the Hopf algebras $(U_w g_{(\kappa_1,0)}, \Delta_{01})$ have classical commutation rules and no r–matrix.

(3) Finally, the same method applied to

$(g_{(\kappa_1,\kappa_2)}, \delta_{12}^{(0,1)})$ gives

$$\begin{aligned} \Delta_{12}(J_{12}) &= 1 \otimes J_{12} + J_{12} \otimes 1, \\ \Delta_{12}(J_{01}) &= e^{-\frac{w}{2}J_{12}} \otimes J_{01} + J_{01} \otimes e^{\frac{w}{2}J_{12}}, \\ \Delta_{12}(J_{02}) &= e^{-\frac{w}{2}J_{12}} \otimes J_{02} + J_{02} \otimes e^{\frac{w}{2}J_{12}}, \end{aligned} \tag{3.17}$$

$$[J_{12}, J_{01}]_{12} = J_{02}, \qquad [J_{12}, J_{02}]_{12} = -\kappa_2 J_{01}, \qquad [J_{01}, J_{02}]_{12} = \kappa_1 \frac{\sinh w J_{12}}{w}. \tag{3.18}$$

The affine contraction $\kappa_1 \to 0$ leads to non-coboundary deformations. In particular, if $\kappa_2 = -1$ we obtain a quantum (1+1) Poincaré algebra analogous to the Vaksman–Korogodsky deformation of the Euclidean case ($\kappa_2 = 1$).

On the other hand, as first order terms within a Hopf algebra deformation, the bialgebras here described give information about the commutation rules in the dual Hopf algebras that define the quantum groups linked with our algebras. If the cocommutator is written as:

$$\delta'(X_\gamma) = f_\gamma^{\alpha\beta} X_\alpha \otimes X_\beta, \tag{3.19}$$

then the dual bracket is (first order in w and linear in the generators):

$$[x^\alpha, x^\beta] = f_\gamma^{\alpha\beta} x^\gamma, \tag{3.20}$$

plus, in general, higher order terms both in w and in the generators. Therefore, these three sets of fundamental bialgebras give rise to bialgebras whose "linearized" dual algebras are characterized by the brackets given in table II. Note that these "first order dual algebras" are solvable ones and, in this case, no dual bracket depends on the contraction parameters. It is worth to recall that, at the classical level, these brackets are the first order terms of the Sklyanin ones (either generated by a given r–matrix or not) that provide the Poisson-Lie structures whose deformation-quantization (see [27]) gives the quantum groups mentioned above.

Table II. Dual commutators of fundamental bialgebras.

Dual bracket	$Fun_w(g_{(\kappa_1,\kappa_2)}, \delta_{01}^{(1,0)})$	$Fun_w(g_{(\kappa_1,\kappa_2)}, \delta_{02}^{(1,1)})$	$Fun_w(g_{(\kappa_1,\kappa_2)}, \delta_{12}^{(0,1)})$
$[j_{12}, j_{01}]$	$w\, j_{12}$	0	$-w\, j_{01}$
$[j_{12}, j_{02}]$	0	$w\, j_{12}$	$-w\, j_{02}$
$[j_{01}, j_{02}]$	$-w\, j_{02}$	$w\, j_{01}$	0

Finally, the existence of a quantum R–matrix for the contracted Hopf algebra can be easily explored by using the LBC approach. If we want $R = 1 + r + \ldots$

to be a solution of the quantum YBE, then r has to be a solution for its classical counterpart: $[r,r]=0$. For instance, it can be checked that the element

$$\tilde{r}_{02} = z\,(J_{12}\wedge J_{01} + i(J_{01}\otimes J_{01} + J_{02}\otimes J_{02} + J_{12}\otimes J_{12})) \tag{3.21}$$

generates the same $so(3)$ coboundary bialgebra as r_{02} and fulfills the classical YBE. In fact, the r–matrix (3.21) is the first order term of the universal R–matrix corresponding to $(U_w so(3), \Delta_{02})$. By applying Φ_2 on it we obtain

$$\tilde{r}'_{02} = \frac{\Phi_2^{-1}(z)}{\kappa_1\kappa_2}(\sqrt{\kappa_1\kappa_2}J_{12}\wedge J_{01} + i(\kappa_2 J_{01}\otimes J_{01} + J_{02}\otimes J_{02} + \kappa_1 J_{12}\otimes J_{12})). \tag{3.22}$$

We see that the LBC's for $(so(3), \delta(\tilde{r}_{02}))$ characterized in table I do not provide well defined expressions for the limits $\kappa_i \to 0$ of (3.22). We would have to take $z = \kappa_1\kappa_2\, w$ in order to get a contraction of $\tilde{r}_{02}$, but in that case both LBC's are not fundamental and give the trivial bialgebra as a contracted structure. This fact is related to the problem of obtaining a universal R–matrix verifying the quantum YBE for the Euclidean or Poincaré algebras by contraction; the LBC analysis shows that, for $e(2)$ or $p(1+1)$ this is not possible.

4 The so(4) case

We now start from the $so(4)$ classical r–matrix given by

$$r_{03,12} = z(\widetilde{J}_{13}\wedge\widetilde{J}_{01} + \widetilde{J}_{23}\wedge\widetilde{J}_{02}). \tag{4.1}$$

The coboundary bialgebra $(so(4), \delta(r_{03,12}))$ is, explicitly,

$$\begin{aligned}
\delta_{03,12}(\widetilde{J}_{03}) &= \delta_{03,12}(\widetilde{J}_{12}) = 0,\\
\delta_{03,12}(\widetilde{J}_{01}) &= z(\widetilde{J}_{01}\wedge\widetilde{J}_{03} + \widetilde{J}_{23}\wedge\widetilde{J}_{12}),\\
\delta_{03,12}(\widetilde{J}_{02}) &= z(\widetilde{J}_{02}\wedge\widetilde{J}_{03} - \widetilde{J}_{13}\wedge\widetilde{J}_{12}),\\
\delta_{03,12}(\widetilde{J}_{13}) &= z(\widetilde{J}_{13}\wedge\widetilde{J}_{03} - \widetilde{J}_{02}\wedge\widetilde{J}_{12}),\\
\delta_{03,12}(\widetilde{J}_{23}) &= z(\widetilde{J}_{23}\wedge\widetilde{J}_{03} + \widetilde{J}_{01}\wedge\widetilde{J}_{12}).
\end{aligned} \tag{4.2}$$

The label $\{03,12\}$ recalls the pair of generators which are primitive, i.e., their cocommutators are zero. The first pair of subscripts indicates the "principal" primitive generator which appears in the term of the cocommutator $\delta(X)$ containing X and the second one labels the "secondary" primitive generator that generates the quantum $so(2)$ subalgebra (now unidimensional) embedded into $so(4)$.

The quantum $so(4)$ algebra linked to this specific bialgebra is given by the following coproducts and deformed commutation rules (omitting the labels $\{03,12\}$) [19]:

$$\Delta(\widetilde{J}_{03}) = 1\otimes\widetilde{J}_{03} + \widetilde{J}_{03}\otimes 1, \qquad \Delta(\widetilde{J}_{12}) = 1\otimes\widetilde{J}_{12} + \widetilde{J}_{12}\otimes 1,$$

$$
\begin{aligned}
\Delta(\widetilde{J}_{01}) &= e^{-\frac{z}{2}\widetilde{J}_{03}}\cosh(\tfrac{z}{2}\widetilde{J}_{12}) \otimes \widetilde{J}_{01} + \widetilde{J}_{01} \otimes e^{\frac{z}{2}\widetilde{J}_{03}}\cosh(\tfrac{z}{2}\widetilde{J}_{12}) \\
&\quad - e^{-\frac{z}{2}\widetilde{J}_{03}}\sinh(\tfrac{z}{2}\widetilde{J}_{12}) \otimes \widetilde{J}_{23} + \widetilde{J}_{23} \otimes e^{\frac{z}{2}\widetilde{J}_{03}}\sinh(\tfrac{z}{2}\widetilde{J}_{12}), \\
\Delta(\widetilde{J}_{02}) &= e^{-\frac{z}{2}\widetilde{J}_{03}}\cosh(\tfrac{z}{2}\widetilde{J}_{12}) \otimes \widetilde{J}_{02} + \widetilde{J}_{02} \otimes e^{\frac{z}{2}\widetilde{J}_{03}}\cosh(\tfrac{z}{2}\widetilde{J}_{12}) \\
&\quad + e^{-\frac{z}{2}\widetilde{J}_{03}}\sinh(\tfrac{z}{2}\widetilde{J}_{12}) \otimes \widetilde{J}_{13} - \widetilde{J}_{13} \otimes e^{\frac{z}{2}\widetilde{J}_{03}}\sinh(\tfrac{z}{2}\widetilde{J}_{12}), \\
\Delta(\widetilde{J}_{13}) &= e^{-\frac{z}{2}\widetilde{J}_{03}}\cosh(\tfrac{z}{2}\widetilde{J}_{12}) \otimes \widetilde{J}_{13} + \widetilde{J}_{13} \otimes e^{\frac{z}{2}\widetilde{J}_{03}}\cosh(\tfrac{z}{2}\widetilde{J}_{12}) \\
&\quad + e^{-\frac{z}{2}\widetilde{J}_{03}}\sinh(\tfrac{z}{2}\widetilde{J}_{12}) \otimes \widetilde{J}_{02} - \widetilde{J}_{02} \otimes e^{\frac{z}{2}\widetilde{J}_{03}}\sinh(\tfrac{z}{2}\widetilde{J}_{12}), \\
\Delta(\widetilde{J}_{23}) &= e^{-\frac{z}{2}\widetilde{J}_{03}}\cosh(\tfrac{z}{2}\widetilde{J}_{12}) \otimes \widetilde{J}_{23} + \widetilde{J}_{23} \otimes e^{\frac{z}{2}\widetilde{J}_{03}}\cosh(\tfrac{z}{2}\widetilde{J}_{12}) \\
&\quad - e^{-\frac{z}{2}\widetilde{J}_{03}}\sinh(\tfrac{z}{2}\widetilde{J}_{12}) \otimes \widetilde{J}_{01} + \widetilde{J}_{01} \otimes e^{\frac{z}{2}\widetilde{J}_{03}}\sinh(\tfrac{z}{2}\widetilde{J}_{12});
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
{}[\widetilde{J}_{13}, \widetilde{J}_{23}] &= \frac{1}{z}\sinh(z\widetilde{J}_{12})\cosh(z\widetilde{J}_{03}), \quad & [\widetilde{J}_{23}, \widetilde{J}_{02}] &= \frac{1}{z}\sinh(\widetilde{J}_{03})\cosh(z\widetilde{J}_{12}), \\
{}[\widetilde{J}_{13}, \widetilde{J}_{01}] &= \frac{1}{z}\sinh(z\widetilde{J}_{03})\cosh(z\widetilde{J}_{12}), \quad & [\widetilde{J}_{01}, \widetilde{J}_{02}] &= \frac{1}{z}\sinh(z\widetilde{J}_{12})\cosh(z\widetilde{J}_{03}).
\end{aligned}
\tag{4.4}
$$

4.1 $so(4)$ bialgebras and their contractions

We can build five more coboundary bialgebras by permutation of the set of indices $\{0,1,2,3\}$. For instance, the transition $r_{03,12} \to r_{01,23}$ can be got by applying on the former r–matrix any of the permutations mapping 03 into 01 and 12 into 23. There are four such permutations with cycle decomposition: $\pi_{(123)}, \pi_{(13)}, \pi_{(0123)}$ and $\pi_{(013)}$; all of them provide the same result. In particular we choose the following representatives for these five structures:

$$
\begin{aligned}
&r_{01,23} = \pi_{(13)}(r_{03,12}), \qquad && r_{02,13} = \pi_{(23)}(r_{03,12}), \qquad && r_{12,03} = \pi_{(01)(23)}(r_{03,12}), \\
&r_{13,02} = \pi_{(01)}(r_{03,12}), \qquad && r_{23,01} = \pi_{(02)}(r_{03,12}).
\end{aligned}
\tag{4.5}
$$

Similarly, we can deduce their corresponding cocommutators.

The formal transformation Φ_3 (2.20) applied to the six r–matrices (4.1,4.5) and to their corresponding cocommutators, gives rise to coboundary Lie bialgebras for the eight CK algebras $g_{(\kappa_1,\kappa_2,\kappa_3)}$ with all $\kappa_i \neq 0$: $so(4)$, $so(3,1)$ (4 times) and $so(2,2)$ (3 times). Explicit expressions are given in Appendix A.

Therefore, it is straightforward to obtain the classification of the LBC's for each one of the three classical contractions ϕ_{ε_i} ($i = 1,2,3$): it suffices to study the properties of the limit where just one κ_i (ε_i) goes to zero while keeping the other two unchanged. Afterwards we have to find for each case, the fundamental contraction constant f_0 (the minimum value of n which guarantees that $\delta' = \lim_{\varepsilon_i \to 0}\delta$ exists), and the coboundary contraction constant c_0 (idem with respect to $r' = \lim_{\varepsilon_i \to 0} r$). We state the final result as:

Theorem 4.1. *Consider the coboundary Lie bialgebra $(g_{(\kappa_1,\kappa_2,\kappa_3)}, \delta'(r'_{ab,cd}))$ $\forall \kappa_i \neq 0$. The fundamental and the coboundary LBC's are defined, in this order, by the contraction constants f_0 and c_0 associated to a given classical contraction ϕ_{ε_i} displayed in table III.*

Table III. LBC constants for $g_{(\kappa_1,\kappa_2,\kappa_3)}$.

Lie bialgebra	ϕ_{ε_1}		ϕ_{ε_2}		ϕ_{ε_3}	
	f_0	c_0	f_0	c_0	f_0	c_0
$(g,\delta'_{01,23})$	1	1	0	2	2	2
$(g,\delta'_{02,13})$	1	1	1	1	2	2
$(g,\delta'_{03,12})$	1	1	1	1	1	1
$(g,\delta'_{12,03})$	2	2	1	1	2	2
$(g,\delta'_{13,02})$	2	2	1	1	1	1
$(g,\delta'_{23,01})$	2	2	0	2	1	1

Before going to the quantum algebras note that, once again, all fundamental LBC's that preserve the deformation parameter ($f_0 = 0$) are not of the coboundary type. In general, non-coboundary bialgebras arise whenever $f_0 < c_0$, this is, when the LBC $(\phi_{\varepsilon_2}, 0)$ is applied on bialgebras that come from $(so(4), \delta(r_{01,23}))$ and $(so(4), \delta(r_{23,01}))$. It is easy to check that, if $\kappa_2 \to 0$, no antisymmetric element of $g \otimes g$ can generate the contracted cocommutators (A.2) or (A.7). This agrees with Zakrzewski's theorem [28] proving that, for $N > 2$, all semidirect sums $t_N \odot so(p', q')$ with $p' + q' = N$ do not admit non-coboundary bialgebras (in our case, the algebras with $\kappa_2 = 0$ do not admit such a kind of semidirect sum decomposition). Among this non-coboundary objects, we can find two different (2+1) Galilean bialgebras: $(g_{(0,0,1)}, \delta^{(1,0,2)}_{01,23})$ and $(g_{(0,0,1)}, \delta^{(2,0,1)}_{23,01})$, that we shall relate in the sequel with two different quantum (2+1) Galilean algebras.

4.2 Hopf algebra contractions of $U_z so(4)$

We have obtained six sets of fundamental bialgebras, and four of them are fundamental coboundary ones. Among them, the Hopf structure of

$(g_{(\kappa_1,\kappa_2,\kappa_3)}, \delta^{(1,1,1)}_{03,12})$ has been studied in [19]; the particular cases $(U_w g_{(-1,-1,1)}, \Delta_{03,12})$ $(so(3,1)_q)$ and

$(U_w g_{(0,1,1)}, \Delta_{03,12})$ $(e(3)_q)$ have been given in [29] and [30], respectively. We shall explicitly discuss two examples containing non-coboundary objects, since they provide new quantizations of some interesting algebras.

• By applying the permutation $\pi_{(13)}$ to relations (4.3–4.4) and the Hopf algebra contractions defined by the set of fundamental LBC's given in the first row of table III, we obtain the Hopf algebra $(U_w g_{(\kappa_1,\kappa_2,\kappa_3)}, \Delta_{01,23})$ (we omit the $\{01, 23\}$ label):

$$
\begin{aligned}
&\Delta(J_{01}) = 1 \otimes J_{01} + J_{01} \otimes 1, \qquad \Delta(J_{23}) = 1 \otimes J_{23} + J_{23} \otimes 1,\\
&\Delta(J_{02}) = e^{-\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23}) \otimes J_{02} + J_{02} \otimes e^{\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23})\\
&\qquad + e^{-\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23}) \otimes \kappa_1 J_{13} - \kappa_1 J_{13} \otimes e^{\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23}),\\
&\Delta(J_{03}) = e^{-\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23}) \otimes J_{03} + J_{03} \otimes e^{\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23})\\
&- e^{-\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23}) \otimes \kappa_1\kappa_3 J_{12} + \kappa_1\kappa_3 J_{12} \otimes e^{\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23}), \qquad (4.6)\\
&\Delta(J_{12}) = e^{-\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23}) \otimes J_{12} + J_{12} \otimes e^{\frac{w}{2}\kappa_3 J_{01}}\, \mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23})
\end{aligned}
$$

$$
\begin{aligned}
&-e^{-\frac{w}{2}\kappa_3 J_{01}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23})\otimes J_{03}+J_{03}\otimes e^{\frac{w}{2}\kappa_3 J_{01}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23}),\\
\Delta(J_{13})&=e^{-\frac{w}{2}\kappa_3 J_{01}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23})\otimes J_{13}+J_{13}\otimes e^{\frac{w}{2}\kappa_3 J_{01}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23})\\
&+e^{-\frac{w}{2}\kappa_3 J_{01}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23})\otimes\kappa_3 J_{02}-\kappa_3 J_{02}\otimes e^{\frac{w}{2}\kappa_3 J_{01}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{23});
\end{aligned}
$$

$$
\begin{aligned}
[J_{02},J_{03}]&=\kappa_1\kappa_2\frac{1}{w}\,\mathrm{S}_{-\kappa_1\kappa_3}(wJ_{23})\,\mathrm{C}_{-\kappa_3^2}(wJ_{01}),\\
[J_{02},J_{12}]&=\kappa_2\frac{1}{w}\,\mathrm{S}_{-\kappa_3^2}(wJ_{01})\,\mathrm{C}_{-\kappa_1\kappa_3}(wJ_{23}),\\
[J_{12},J_{13}]&=\kappa_2\frac{1}{w}\,\mathrm{S}_{-\kappa_1\kappa_3}(wJ_{23})\,\mathrm{C}_{-\kappa_3^2}(wJ_{01}),\\
[J_{03},J_{13}]&=\kappa_2\kappa_3\frac{1}{w}\,\mathrm{S}_{-\kappa_3^2}(wJ_{01})\,\mathrm{C}_{-\kappa_1\kappa_3}(wJ_{23}).
\end{aligned}
\tag{4.7}
$$

We recall [18] that

$$
\mathrm{C}_{-\kappa}(x):=\frac{e^{\sqrt{\kappa}x}+e^{-\sqrt{\kappa}x}}{2},\qquad \mathrm{S}_{-\kappa}(x):=\frac{e^{\sqrt{\kappa}x}-e^{-\sqrt{\kappa}x}}{2\sqrt{\kappa}}.
\tag{4.8}
$$

It is important to point out that in this case the contractions $\kappa_2\to 0$ or $\kappa_3\to 0$ give a deformed coproduct with classical commutation rules. We shall insist in this point later.

- The Hopf algebra $(U_w g_{(\kappa_1,\kappa_2,\kappa_3)},\Delta_{23,01})$ is given (omitting the label again) by

$$
\begin{aligned}
\Delta(J_{23})&=1\otimes J_{23}+J_{23}\otimes 1,\qquad \Delta(J_{01})=1\otimes J_{01}+J_{01}\otimes 1,\\
\Delta(J_{02})&=e^{-\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\otimes J_{02}+J_{02}\otimes e^{\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\\
&+e^{-\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\otimes\kappa_1 J_{13}-\kappa_1 J_{13}\otimes e^{\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01}),\\
\Delta(J_{03})&=e^{-\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\otimes J_{03}+J_{03}\otimes e^{\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\\
&-e^{-\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\otimes\kappa_1\kappa_3 J_{12}+\kappa_1\kappa_3 J_{12}\otimes e^{\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01}),\\
\Delta(J_{12})&=e^{-\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\otimes J_{12}+J_{12}\otimes e^{\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\\
&-e^{-\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\otimes J_{03}+J_{03}\otimes e^{\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01}),\\
\Delta(J_{13})&=e^{-\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\otimes J_{13}+J_{13}\otimes e^{\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{C}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\\
&+e^{-\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01})\otimes\kappa_3 J_{02}-\kappa_3 J_{02}\otimes e^{\frac{w}{2}\kappa_1 J_{23}}\,\mathrm{S}_{-\kappa_1\kappa_3}(\tfrac{w}{2}J_{01});
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
[J_{02},J_{03}]&=\kappa_1\kappa_2\frac{1}{w}\,\mathrm{S}_{-\kappa_1^2}(wJ_{23})\,\mathrm{C}_{-\kappa_1\kappa_3}(wJ_{01}),\\
[J_{02},J_{12}]&=\kappa_2\frac{1}{w}\,\mathrm{S}_{-\kappa_1\kappa_3}(wJ_{01})\,\mathrm{C}_{-\kappa_1^2}(wJ_{23}),\\
[J_{12},J_{13}]&=\kappa_2\frac{1}{w}\,\mathrm{S}_{-\kappa_1^2}(wJ_{23})\,\mathrm{C}_{-\kappa_1\kappa_3}(wJ_{01}),\\
[J_{03},J_{13}]&=\kappa_2\kappa_3\frac{1}{w}\,\mathrm{S}_{-\kappa_1\kappa_3}(wJ_{01})\,\mathrm{C}_{-\kappa_1^2}(wJ_{23}).
\end{aligned}
\tag{4.10}
$$

Similarly to the previous example, now the coproduct is invariant under the $\kappa_2\to 0$ limit, and the contractions

$\kappa_1\to 0$ or $\kappa_2\to 0$ provide classical commutation rules. This case can be easily related to the previous one can by interchanging the role of the contractions ϕ_{ε_1} and ϕ_{ε_3}.

4.3 Quantum (2+1) Poincaré and Galilei algebras

The "geometrical" orthogonal basis we have been working with can be interpreted as a kinematical one. In this way we can explore the differences underlying these contractions from a physical point of view. In particular, by assuming the standard kinematical assignation [19]:

$$J_{01} = H, \quad J_{02} = P_1, \quad J_{03} = P_2, \quad J_{12} = K_1, \quad J_{13} = K_2, \quad J_{23} = J, \tag{4.11}$$

that expresses in a physical basis the Poincaré algebra $g_{(0,-1,1)}$, we obtain six different deformations (all of coboundary type since $f_0 = c_0$ in all the cases for the ϕ_{ε_1} contraction) that can be splitted into two classes:

(1) *Quantum (2+1) Poincaré algebras with deformed commutation rules*: the case $(U_w g_{(0,-1,1)}, \Delta_{01,23})$ is the three-dimensional analogue to the κ–Poincaré algebra [31]. Both $(U_w g_{(0,-1,1)}, \Delta_{02,13})$ and $(U_w g_{(0,-1,1)}, \Delta_{03,12})$ were studied in [19], and can be interpreted as relativistic symmetries with a discretized spatial direction.

(2) *Twisted (2+1) Poincaré algebras*: the contraction $\kappa_1 \to 0$ turns commutators of the deformations $(U_w g_{(0,-1,1)}, \Delta_{12,03})$, $(U_w g_{(0,-1,1)}, \Delta_{13,02})$ and $(U_w g_{(0,-1,1)}, \Delta_{23,01})$ into classical ones (see (4.10)). Moreover, the final coproducts contain only first order deformations, and four generators are now primitive (see (4.9)). As a consequence, these quantum algebras can be shown to be twistings of the classical structure where the twisting operator is just the exponential of the classical r–matrix (compare with [32]). These three structures appear for a transformation law $z = \kappa_1 w$ of the deformation parameter under the affine contraction, and to our knowdledge, have not been so far considered in the literature. Among them, the quantum algebra $(U_w g_{(0,-1,1)}, \Delta_{23,01})$ presents interesting properties. Its coproduct contains the boosts as the non-primitive generators, in the form

$$\begin{aligned} \Delta_{23,01}(K_1) &= 1 \otimes K_1 + K_1 \otimes 1 - \frac{w}{2} H \wedge P_2, \\ \Delta_{23,01}(K_2) &= 1 \otimes K_2 + K_2 \otimes 1 + \frac{w}{2} H \wedge P_1, \end{aligned} \tag{4.12}$$

and therefore is a deformation of the (2+1) Poincaré algebra with classical commutation rules, deformed boosts and rotational symmetry preserved. All these properties were required in [33] as properties for physically meaningful space-time deformations of Poincaré algebra.

As far as the (2+1) Galilei algebra $g_{(0,0,1)}$ is concerned, some new features appear. We shall have six deformations obtained from the Poincaré ones by using the LBC's defined by ϕ_{ε_2}. This contraction splits into three classes the structures so obtained (the kinematical assignation (4.11) remains the same):

(1) *Quantum (2+1) Galilei algebras with deformed commutation rules*: there are two cases, $(U_w g_{(0,0,1)}, \Delta_{02,13})$ and $(U_w g_{(0,0,1)}, \Delta_{03,12})$. These coboundary deformations are connected with the first class of quantum Poincaré algebras and were studied in [19].

(2) *Twisted (2+1) Galilei algebras*: the quantum algebras $(U_w g_{(0,0,1)}, \Delta_{12,03})$ and $(U_w g_{(0,0,1)}, \Delta_{13,02})$ are coboundary deformations with classical commutation rules. The twisting operator is given by the r–matrix.

(3) *Non-coboundary (2+1) Galilei algebras*: the deformations $(U_w g_{(0,0,1)}, \Delta_{01,23})$ and $(U_w g_{(0,0,1)}, \Delta_{23,01})$ have non-deformed commutation rules. The first one has a deformed coproduct with exponential terms (it is just the (2+1) analogue of the deformation given in [34]). The latter presents a simpler deformation (with only linear non-primitive terms) and reproduces the deformation of boosts (4.12) in the non-relativistic case.

In this way, we can see how the different permutations of $so(4)$ bialgebras give rise to non-equivalent deformations as long as LBC's are applied in a systematic way. Finally, let us note that if we construct (as in table II) the linear part of the dual commutation rules among the generators of the dual basis with respect to a fixed set of fundamental bialgebras, we would find that here some coefficients κ_i (further to the contracted deformation parameter w) would appear as structure constants of the (solvable) dual Lie algebras; therefore, cocommutators are not invariant under fundamental contractions. However, it is easy to prove that the fundamental character of the LBC's will ensure that the Abelian algebra does not appear within this dual structure.

5 The so(5) case

We summarize the results obtained when this method is applied on the $so(5)$ coboundary bialgebra generated by

$$r_{04,13} = z(\widetilde{J}_{14} \wedge \widetilde{J}_{01} + \widetilde{J}_{24} \wedge \widetilde{J}_{02} + \widetilde{J}_{34} \wedge \widetilde{J}_{03} + \widetilde{J}_{23} \wedge \widetilde{J}_{12}). \tag{5.1}$$

A straightforward computation gives rise to the following explicit form of the cocommutator $\delta_{04,13}$:

$$\begin{aligned}
&\delta_{04,13}(\widetilde{J}_{04}) = 0, \qquad \delta_{04,13}(\widetilde{J}_{12}) = z(\widetilde{J}_{12} \wedge \widetilde{J}_{13}),\\
&\delta_{04,13}(\widetilde{J}_{13}) = 0, \qquad \delta_{04,13}(\widetilde{J}_{23}) = z(\widetilde{J}_{23} \wedge \widetilde{J}_{13}),\\
&\delta_{04,13}(\widetilde{J}_{02}) = z(\widetilde{J}_{02} \wedge \widetilde{J}_{04} + \widetilde{J}_{12} \wedge \widetilde{J}_{14} + \widetilde{J}_{34} \wedge \widetilde{J}_{23} + \widetilde{J}_{23} \wedge \widetilde{J}_{01} + \widetilde{J}_{12} \wedge \widetilde{J}_{03}),\\
&\delta_{04,13}(\widetilde{J}_{24}) = z(\widetilde{J}_{24} \wedge \widetilde{J}_{04} + \widetilde{J}_{23} \wedge \widetilde{J}_{03} + \widetilde{J}_{01} \wedge \widetilde{J}_{12} + \widetilde{J}_{12} \wedge \widetilde{J}_{34} + \widetilde{J}_{23} \wedge \widetilde{J}_{14}),\\
&\delta_{04,13}(\widetilde{J}_{01}) = z(\widetilde{J}_{01} \wedge \widetilde{J}_{04} + \widetilde{J}_{24} \wedge \widetilde{J}_{12} + \widetilde{J}_{34} \wedge \widetilde{J}_{13} + \widetilde{J}_{02} \wedge \widetilde{J}_{23}),\\
&\delta_{04,13}(\widetilde{J}_{34}) = z(\widetilde{J}_{34} \wedge \widetilde{J}_{04} + \widetilde{J}_{02} \wedge \widetilde{J}_{23} + \widetilde{J}_{01} \wedge \widetilde{J}_{13} + \widetilde{J}_{24} \wedge \widetilde{J}_{12}),\\
&\delta_{04,13}(\widetilde{J}_{03}) = z(\widetilde{J}_{03} \wedge \widetilde{J}_{04} + \widetilde{J}_{13} \wedge \widetilde{J}_{14} + \widetilde{J}_{23} \wedge \widetilde{J}_{24} + \widetilde{J}_{02} \wedge \widetilde{J}_{12}),\\
&\delta_{04,13}(\widetilde{J}_{14}) = z(\widetilde{J}_{14} \wedge \widetilde{J}_{04} + \widetilde{J}_{13} \wedge \widetilde{J}_{03} + \widetilde{J}_{12} \wedge \widetilde{J}_{02} + \widetilde{J}_{24} \wedge \widetilde{J}_{23}).
\end{aligned} \tag{5.2}$$

Note that, in this case, the term $\widetilde{J}_{23} \wedge \widetilde{J}_{12}$ in (5.1) corresponds to the Lie bialgebra $(so(3), \delta(r_{13}))$ embedded into $(so(5), \delta(r_{04,13}))$. This Lie subalgebra is generated by $\langle \widetilde{J}_{12}, \widetilde{J}_{13}, \widetilde{J}_{23} \rangle$ with $\widetilde{J}_{13}$ playing the role of a secondary primitive generator.

The 120 elements of the permutation group S_5 on the five indices $\{0, 1, 2, 3, 4\}$ can be casted into 30 different classes attending to their action on the ordered set of two primitive generators; each of these classes includes four permutations giving

rise to the tetrads $(ab,cd),(ba,cd),(ab,dc)$ and (ba,dc) starting from (ab,cd). The four permutations in each class would lead to the same r–matrix starting from $r_{04,13}$. Thus, the mapping Φ_4 (2.20) leads to 30 Lie bialgebra structures for the pseudo-orthogonal algebras included in the CK algebra $g_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)}$. The complete classification of its LBC's is given by the following theorem.

Theorem 5.1. *LBC's of the bialgebra $(g_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)},\delta'(r'_{ab,cd}))$ are classified by the contraction constants given in table IV.*

The explicit form of the $so(5)$ quantization corresponding to the Lie bialgebra (5.2) is rather complicated, and has been studied in [35]. However, once the affine contraction $\kappa_1 \to 0$ is carried out, the structure is much simpler as it can be seen, for instance, in the Hopf algebra that quantizes the coboundary Lie bialgebra $(g_{(0,\kappa_2,\kappa_3,\kappa_4)},\delta^{(1,1,1,1)}_{04,13})$ which has been developed in [13]. Another interesting examples are the κ–Poincaré algebra [31] which appears as a quantization of $(g_{(0,-1,1,1)},\delta^{(1,2,2,2)}_{01,34})$, and the Giller's (3+1) Galilei deformation [34] obtained by applying on the former the ϕ_{ε_2} contraction; note that the latter it is not a coboundary (see third row of table IV).

These kinematical realizations of the CK algebras are obtained provided a precise physical meaning for the generators J_{ab} has been considered (see [13]). This fact gives sense to the exhaustive classification of LBC's summarized in the Theorem 5.1. Although table IV defines 30 sets of quantum CK algebras $(g_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)},\Delta_{ab,cd})$, their explicit expressions are quite cumbersome to obtain. However, first order deformations are available and provide essential information to characterize the deformations. If considered as interesting after this primary analysis, the whole quantum structure can be straightforwardly derived by permutation and contraction from the $so(5)$ structure, already known.

A glimpse on Table IV reveals that the ratio of fundamental LBC's leading to divergencies in the r–matrix ($f_0 < c_0$) has diminished strongly with respect to the lower dimensional cases (no ϕ_{ε_1} and ϕ_{ε_4} LBC's give such a problem). In fact, fundamental coboundary contractions ($f_0 = c_0$) are the most frequent objects we obtain, and the contraction constant $f_0 = 2$ plays now a prominent role. No twisted (3+1) Poincaré algebra preserving rotational symmetry (similar to (4.12)) seems to be available now.

Table IV. LBC constants for $g_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)}$.

Lie bialgebra	ϕ_{ε_1}		ϕ_{ε_2}		ϕ_{ε_3}		ϕ_{ε_4}	
	f_0	c_0	f_0	c_0	f_0	c_0	f_0	c_0
$(g,\delta'_{01,23})$	1	1	0	2	2	2	2	2
$(g,\delta'_{01,24})$	1	1	0	2	2	2	2	2
$(g,\delta'_{01,34})$	1	1	0	2	2	2	2	2
$(g,\delta'_{02,13})$	1	1	1	1	2	2	2	2
$(g,\delta'_{02,14})$	1	1	1	1	2	2	2	2
$(g,\delta'_{02,34})$	1	1	2	2	2	2	2	2
$(g,\delta'_{12,03})$	2	2	1	1	2	2	2	2
$(g,\delta'_{12,04})$	2	2	1	1	2	2	2	2
$(g,\delta'_{12,34})$	2	2	2	2	2	2	2	2
$(g,\delta'_{03,12})$	1	1	1	1	2	2	2	2
$(g,\delta'_{03,14})$	1	1	1	1	1	1	2	2
$(g,\delta'_{03,24})$	1	1	2	2	1	1	2	2
$(g,\delta'_{13,02})$	2	2	1	1	2	2	2	2
$(g,\delta'_{13,04})$	2	2	1	1	1	1	2	2
$(g,\delta'_{13,24})$	2	2	2	2	1	1	2	2
$(g,\delta'_{23,01})$	2	2	2	2	2	2	2	2
$(g,\delta'_{23,04})$	2	2	2	2	1	1	2	2
$(g,\delta'_{23,14})$	2	2	2	2	1	1	2	2
$(g,\delta'_{04,12})$	1	1	1	1	2	2	1	1
$(g,\delta'_{04,13})$	1	1	1	1	1	1	1	1
$(g,\delta'_{04,23})$	1	1	2	2	1	1	1	1
$(g,\delta'_{14,02})$	2	2	1	1	2	2	1	1
$(g,\delta'_{14,03})$	2	2	1	1	1	1	1	1
$(g,\delta'_{14,23})$	2	2	2	2	1	1	1	1
$(g,\delta'_{24,01})$	2	2	2	2	2	2	1	1
$(g,\delta'_{24,03})$	2	2	2	2	1	1	1	1
$(g,\delta'_{24,13})$	2	2	2	2	1	1	1	1
$(g,\delta'_{34,01})$	2	2	2	2	0	2	1	1
$(g,\delta'_{34,02})$	2	2	2	2	0	2	1	1
$(g,\delta'_{34,12})$	2	2	2	2	0	2	1	1

6 Concluding remarks

We have presented a general framework to obtain and classify Lie bialgebra structures by contraction. Due to its physical interest, we have developed in a explicit way the case of quasi-orthogonal algebras endowed with cocommutators that support the uniparametric Drinfel'd–Jimbo deformation for these algebras. However,

this general theory can be straightforwardly applied to any other Lie bialgebra.

In order to summarize the information we have obtained in the previous sections, it is interesting to emphasize the link between some quantum algebras found in the literature and the underlying Lie bialgebras we have obtained by contraction. We display in the following table explicit references together with the notation for the fundamental bialgebra $(g_{(\kappa_1,\dots\kappa_N)}, \delta_{ab,cd}^{(f_{01},\dots,f_{0N})})$ we have used in this paper (we omit the constants f_{0i}); for the corresponding quantum algebras we give a single reference, no exhaustiveness is attempted here.

Fundamental bialgebra	Quantum algebra	References
$(g_{(0,1)}, \delta_{12})$	$e(2)_q$	[21]
$(g_{(0,1)}, \delta_{02})$	$e(2)_q$	[24]
$(g_{(0,-1)}, \delta_{02})$	$p(1+1)_q$	[25]
$(g_{(0,0)}, \delta_{02})$	$g(1+1)_q$	[26]
$(g_{(0,1,1)}, \delta_{03,12})$	$e(3)_q$	[30]
$(g_{(0,-1,1)}, \delta_{01,23})$	$p(2+1)_q$	[34]
$(g_{(0,-1,1)}, \delta_{03,12})$	$p(2+1)_q$	[19]
$(g_{(0,0,1)}, \delta_{01,23})$	$g(2+1)_q$	[34]
$(g_{(0,0,1)}, \delta_{03,12})$	$g(2+1)_q$	[19]
$(g_{(0,1,1,1)}, \delta_{04,13})$	$e(4)_q$	[13]
$(g_{(0,-1,1,1)}, \delta_{01,23})$	$p(3+1)_q$	[31]
$(g_{(0,-1,1,1)}, \delta_{04,13})$	$p(3+1)_q$	[13]
$(g_{(0,0,1,1)}, \delta_{01,23})$	$g(3+1)_q$	[34]
$(g_{(0,0,1,1)}, \delta_{04,13})$	$g(3+1)_q$	[13]

One of the advantages of this systematic approach is the way in which non-coboundary structures appear (they are natural consequences of exhausting all contraction possibilities). In general, the obtention of Lie bialgebras corresponding to non-semisimple groups is far from being simple, and the application of LBC's seems to provide a relevant subset among them. In this classification context, twisted structures appear quite frequently under contraction.

Another interesting field of applications for this method is the contraction of classical Poisson–Lie structures. A first example in this direction has been given in [36]. The multiplicity of Lie bialgebra structures found in this paper can be translated into Poisson–Lie terms. The study of LBC's could serve as a useful tool in order to classify this kind of Hamiltonian structures. Work on this line is currently in progress.

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Appendix A: Coboundary Lie bialgebras for $g_{(\kappa_1,\kappa_2,\kappa_3)}$

The six coboundary Lie bialgebras for the semi-simple CK algebras $g_{(\kappa_1,\kappa_2,\kappa_3)}$ with all $\kappa_i \neq 0$ are obtained by applying the transformation Φ_3 to the six r–matrices (4.1,4.5) and to their corresponding cocommutators. The analysis of the transformation of the deformation parameter $z = \Phi_3^{-1}(w)$ provides the fundamental and coboundary contraction constants displayed in table III.

For instance, consider the bialgebra generated by the r–matrix:

$$r'_{01,23} = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1}\kappa_2\kappa_3}(J_{03} \wedge J_{13} + \kappa_3 J_{02} \wedge J_{12}). \tag{A.1}$$

The cocommutators are:

$$\begin{aligned}
&\delta'_{01,23}(J_{01}) = \delta'_{01,23}(J_{23}) = 0,\\
&\delta'_{01,23}(J_{02}) = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1}\kappa_3}(\kappa_3 J_{02} \wedge J_{01} - \kappa_1 J_{13} \wedge J_{23}),\\
&\delta'_{01,23}(J_{03}) = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1}\kappa_3}(\kappa_3 J_{03} \wedge J_{01} + \kappa_1\kappa_3 J_{12} \wedge J_{23}),\\
&\delta'_{01,23}(J_{12}) = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1}\kappa_3}(\kappa_3 J_{12} \wedge J_{01} + J_{03} \wedge J_{23}),\\
&\delta'_{01,23}(J_{13}) = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1}\kappa_3}(\kappa_3 J_{13} \wedge J_{01} - \kappa_3 J_{02} \wedge J_{23}).
\end{aligned} \tag{A.2}$$

We find that the transformations of the deformation parameter are $z = \sqrt{\kappa_1}\kappa_2\kappa_3\, w$ for the r–matrix and $z = \sqrt{\kappa_1}\kappa_3\, w$ for the cocommutators. Hence, attending to the contraction factors ε_i, the coboundary contraction constants are $c_0 = \{1, 2, 2\}$ while the fundamental contraction constants are $f_0 = \{1, 0, 2\}$ (we follow the ordering $\{\phi_{\varepsilon_1}, \phi_{\varepsilon_2}, \phi_{\varepsilon_3}\}$).

The remaining structures are:

- Bialgebra generated by $r'_{02,13} = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2}\kappa_3}(\kappa_3 J_{12} \wedge J_{01} + J_{03} \wedge J_{23})$:

$$\begin{aligned}
&\delta'_{02,13}(J_{02}) = \delta'_{02,13}(J_{13}) = 0,\\
&\delta'_{02,13}(J_{01}) = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2}\kappa_3}(\kappa_3 J_{01} \wedge J_{02} - \kappa_1 J_{23} \wedge J_{13}),\\
&\delta'_{02,13}(J_{03}) = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2}\kappa_3}(\kappa_3 J_{03} \wedge J_{02} - \kappa_1\kappa_3 J_{12} \wedge J_{13}),
\end{aligned} \tag{A.3}$$

$$\delta'_{02,13}(J_{12}) = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2}\kappa_3}(\kappa_3 J_{12}\wedge J_{02} - J_{03}\wedge J_{13}),$$
$$\delta'_{02,13}(J_{23}) = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2}\kappa_3}(\kappa_3 J_{23}\wedge J_{02} - \kappa_3 J_{01}\wedge J_{13}).$$

- Bialgebra generated by $r'_{03,12} = \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2\kappa_3}}(J_{13}\wedge J_{01} + J_{23}\wedge J_{02})$:

$$\begin{aligned}
\delta'_{03,12}(J_{03}) &= \delta'_{03,12}(J_{12}) = 0,\\
\delta'_{03,12}(J_{01}) &= \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2\kappa_3}}(J_{01}\wedge J_{03} + \kappa_1 J_{23}\wedge J_{12}),\\
\delta'_{03,12}(J_{02}) &= \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2\kappa_3}}(J_{02}\wedge J_{03} - \kappa_1 J_{13}\wedge J_{12}),\\
\delta'_{03,12}(J_{13}) &= \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2\kappa_3}}(J_{13}\wedge J_{03} - \kappa_3 J_{02}\wedge J_{12}),\\
\delta'_{03,12}(J_{23}) &= \frac{\Phi_3^{-1}(w)}{\sqrt{\kappa_1\kappa_2\kappa_3}}(J_{23}\wedge J_{03} + \kappa_3 J_{01}\wedge J_{12}).
\end{aligned}\tag{A.4}$$

- Bialgebra generated by
$r'_{12,03} = \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2}\kappa_3}(\kappa_3 J_{01}\wedge J_{02} + \kappa_1 J_{13}\wedge J_{23})$:

$$\begin{aligned}
\delta'_{12,03}(J_{03}) &= \delta'_{12,03}(J_{12}) = 0,\\
\delta'_{12,03}(J_{01}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2}\kappa_3}(\kappa_1\kappa_3 J_{01}\wedge J_{12} + \kappa_1 J_{23}\wedge J_{03}),\\
\delta'_{12,03}(J_{02}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2}\kappa_3}(\kappa_1\kappa_3 J_{02}\wedge J_{12} - \kappa_1 J_{13}\wedge J_{03}),\\
\delta'_{12,03}(J_{13}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2}\kappa_3}(\kappa_1\kappa_3 J_{13}\wedge J_{12} - \kappa_3 J_{02}\wedge J_{03}),\\
\delta'_{12,03}(J_{23}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2}\kappa_3}(\kappa_1\kappa_3 J_{23}\wedge J_{12} + \kappa_3 J_{01}\wedge J_{03}).
\end{aligned}\tag{A.5}$$

- Bialgebra generated by $r'_{13,02} = \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2\kappa_3}}(J_{01}\wedge J_{03} + \kappa_1 J_{23}\wedge J_{12})$:

$$\begin{aligned}
\delta'_{13,02}(J_{13}) &= \delta'_{13,02}(J_{02}) = 0,\\
\delta'_{13,02}(J_{01}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2\kappa_3}}(\kappa_1 J_{01}\wedge J_{13} - \kappa_1 J_{23}\wedge J_{02}),\\
\delta'_{13,02}(J_{03}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2\kappa_3}}(\kappa_1 J_{03}\wedge J_{13} - \kappa_1\kappa_3 J_{12}\wedge J_{02}),\\
\delta'_{13,02}(J_{12}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2\kappa_3}}(\kappa_1 J_{12}\wedge J_{13} - J_{03}\wedge J_{02}),\\
\delta'_{13,02}(J_{23}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_2\kappa_3}}(\kappa_1 J_{23}\wedge J_{13} - \kappa_3 J_{01}\wedge J_{02}).
\end{aligned}\tag{A.6}$$

• Bialgebra generated by $r'_{23,01} = \frac{\Phi_3^{-1}(w)}{\kappa_1\kappa_2\sqrt{\kappa_3}}(\kappa_1 J_{12} \wedge J_{13} + J_{02} \wedge J_{03})$:

$$
\begin{aligned}
\delta'_{23,01}(J_{23}) &= \delta'_{23,01}(J_{01}) = 0, \\
\delta'_{23,01}(J_{02}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_3}}(\kappa_1 J_{02} \wedge J_{23} - \kappa_1 J_{13} \wedge J_{01}), \\
\delta'_{23,01}(J_{03}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_3}}(\kappa_1 J_{03} \wedge J_{23} + \kappa_1\kappa_3 J_{12} \wedge J_{01}), \\
\delta'_{23,01}(J_{12}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_3}}(\kappa_1 J_{12} \wedge J_{23} + J_{03} \wedge J_{01}), \\
\delta'_{23,01}(J_{13}) &= \frac{\Phi_3^{-1}(w)}{\kappa_1\sqrt{\kappa_3}}(\kappa_1 J_{13} \wedge J_{23} - \kappa_3 J_{02} \wedge J_{01}).
\end{aligned}
\tag{A.7}
$$

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